

# NEIGHBORLY CUBICAL POLYTOPES AND SPHERES\*

BY

MICHAEL JOSWIG

*Fachbereich Mathematik, AG 7, TU Darmstadt  
64289 Darmstadt, Germany  
e-mail: joswig@mathematik.tu-darmstadt.de*

AND

THILO RÖRIG

*Institut für Mathematik, MA 6-2, TU Berlin  
10623 Berlin, Germany  
e-mail: thilosch@math.tu-berlin.de*

ABSTRACT

We prove that the neighborly cubical polytopes studied by Günter M. Ziegler and the first named author [14] arise as a special case of the neighborly cubical spheres constructed by Babson, Billera and Chan [4]. By relating the two constructions we obtain an explicit description of a non-polytopal neighborly cubical sphere and, further, a new proof of the fact that the cubical equivelar surfaces of McMullen, Schulz and Wills [16] can be embedded into  $\mathbb{R}^3$ .

## 1. Introduction

Our point of departure is a paper by Babson, Billera and Chan [4], in which the authors introduce an inductive construction of cubical  $d$ -spheres from certain sequences of simplicial  $(d - 1)$ -balls and their boundary spheres of dimension  $d - 2$ . It turns out that such cubical spheres reflect many properties of the simplicial spheres involved, but in a cubical disguise. In particular, this way the boundary of a neighborly simplicial  $(d - 1)$ -polytope yields a neighborly cubical

---

\* Both authors are supported by Deutsche Forschungsgemeinschaft, DFG Research Group “Polyhedral Surfaces.”

Received March 15, 2005 and in revised form November 4, 2005

$d$ -sphere, that is, a sphere which has the same  $\lfloor (d - 1)/2 \rfloor$ -skeleton as some high-dimensional cube. Later it was shown by entirely different methods that there even exist neighborly cubical spheres which are polytopal [14].

Our first result, Theorem 3.2, establishes a non-recursive combinatorial description of the cubical spheres studied by Babson, Billera and Chan. From this it can be inferred that particular sequences of pulling triangulations of cyclic polytopes yield the polytopal spheres studied in [14], see Corollary 3.7. As a further benefit of this direct description we observe in Theorem 3.9 that the simplicial spheres involved in the construction must necessarily be polytopal in order to yield polytopal cubical spheres. In Theorem 3.10 we derive that there exists a non-polytopal neighborly cubical 5-sphere on 2048 vertices. Our proof is based on an explicit construction involving a certain non-polytopal 3-sphere  $\mathcal{M}_{425}^{10}$ . This sphere was found by Altshuler [2], and its non-polytopality was verified by Bokowski and Garms [7].

A seemingly different topic is the construction of polyhedral surfaces of ‘unusually large genus,’ pioneered by Coxeter [9] and Ringel [20], and continued by McMullen, Schulz and Wills [16, 17]. Yet we can show that the neighborly cubical 4-polytopes of [14] contain cubical surfaces with  $n$  vertices of genus  $\mathcal{O}(n \log n)$  in their boundary, answering a question of Günter M. Ziegler [24]. Via Schlegel diagrams, that is, perspective projections onto a facet, this gives a simple new proof of the known fact that there are such surfaces which can be embedded into  $\mathbb{R}^3$  with straight faces.

## 2. Cubical and Simplicial Complexes

We review the ingredients of the construction of Babson, Billera, and Chan [4] with slight generalizations.

The reader is advised to consult the monographs [12] and [25] for general information about convex polytopes and related topics.

2.1. CUBES. Consider the  $2d$  affine halfspaces

$$H_i^+ = \{x \in \mathbb{R}^d: x_i \leq 1\} \quad \text{and} \quad H_i^- = \{x \in \mathbb{R}^d: -x_i \leq 1\},$$

which define the  $d$ -dimensional cube

$$C_d = H_1^+ \cap \dots \cap H_d^+ \cap H_1^- \cap \dots \cap H_d^-.$$

The intersections

$$F_i^\sigma = \partial H_i^\sigma \cap C_d = \{x \in C_d: \sigma x_i = 1\}, \quad \sigma \in \{\pm 1\},$$

of their boundaries with the cube are precisely the **facets** of  $C_d$ , that is, its maximal proper faces. The **vertices** of  $C_d$  are all the  $2^d$  vectors of length  $d$  with coordinates  $\pm 1$ , and their joint convex hull is the cube  $C_d$ . Two facets  $F_i^+$  and  $F_i^-$  are parallel, and hence they do not share any vertex of  $C_d$ .

It is a general fact about convex polytopes that each **proper face**, that is, any intersection of the polytope with a supporting hyperplane, can be written as the intersection of facets. Thus, for the special case of the  $d$ -cube, each non-empty face  $F$  can be written uniquely as  $F = F_{i_1}^{\sigma_{i_1}} \cap \dots \cap F_{i_k}^{\sigma_{i_k}}$ , where  $i_j \neq i_l$  for any  $j \neq l$ . By letting  $\sigma_j = 0$  for all  $j \notin \{i_1, \dots, i_k\}$  the non-empty face  $F$  can be identified with the ordered sequence  $(\sigma_1, \dots, \sigma_d)$  of signs  $+1, 0, -1$  of length  $d$ . Conversely, each such sign vector defines a face. For ease of notation we often omit the 1's of  $\pm 1$ .

The intersection of a  $k$ -dimensional cube face, or  $k$ -**face** for short, with a facet is either empty or a  $(k - 1)$ -face. This readily implies that the dimension of the face  $(\sigma_1, \dots, \sigma_d)$  equals the number of 0-entries in its sign vector.

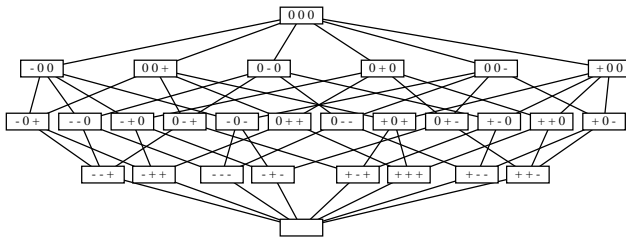


Figure 1. Face lattice of  $C_3$ .

The reflections at the coordinate hyperplanes  $\{x \in \mathbb{R}^d: x_i = 0\}$  generate an elementary abelian group  $\Sigma_d$  of order  $2^d$  which acts sharply transitively on the vertices of  $C_d$ . The full automorphism group  $\Gamma_d$  of the  $d$ -cube is isomorphic to a semi-direct product of  $\Sigma_d$  and the stabilizer of a vertex, which is  $\text{Sym}_d$ , the symmetric group of degree  $d$ . In fact,  $\Gamma_d$  is the wreath product  $\mathbb{Z}_2 \wr \text{Sym}_d$ .

**2.2. REGULAR CELL COMPLEXES AND POSETS.** A **regular cell complex** is a family  $\mathcal{C}$  of closed balls in a Hausdorff space  $X_{\mathcal{C}}$ , such that the interiors of the balls partition  $X_{\mathcal{C}}$  and the boundary of each ball is the union of balls in  $\mathcal{C}$ . The topology of a regular cell complex is completely determined by its face poset; see Björner [6, Prop. 3.1].

An **abstract  $d$ -complex**  $\mathcal{P}$  is a ranked poset of rank  $d + 1$  such that there exist unique lower bounds for any set of elements (that is,  $\mathcal{P}$  is a meet semi-lattice) and every order ideal is combinatorially isomorphic to the face lattice of a polytope. The elements of this partially ordered set are called **faces**. The **boundary** of a finite abstract  $d$ -complex is the set of faces of corank 1 contained in only one maximal face. An abstract  $d$ -complex is a **cubical (simplicial) complex** if every face of  $\mathcal{P}$  is combinatorially isomorphic to a cube (simplex). These are CW posets of polyhedral type as introduced by Björner [6]. Thus we are able to speak of topological properties of abstract  $d$ -complexes. An abstract 2-complex representing a connected 2-manifold without boundary is a **polyhedral surface**.

2.3. MIRRORING. An abstract simplicial complex  $\Delta$  on  $d$  vertices can be seen as a subcomplex of the  $(d - 1)$ -dimensional simplex. The cube  $C_d$  is a **simple  $d$ -polytope**, that is, each of its vertex figures is a  $(d - 1)$ -simplex. Here the **vertex figure** of a polytope  $P$  at a vertex  $v$  is the intersection of  $P$  with an affine hyperplane which separates  $v$  from the other vertices of  $P$ . We construct a (cubical) subcomplex of  $C_d$  which corresponds to a simultaneous embedding of  $\Delta$  into the vertex figures of all the vertices of  $C_d$  such that these embeddings are invariant under the action of the group  $\Sigma_d$ .

Following Babson, Billera and Chan [4] we encode  $\Delta$  in a non-standard way: If  $1, 2, \dots, d$  are the vertices of  $\Delta$  we associate with a face  $\varphi \in \Delta$  the characteristic function of its complement  $\{1, \dots, d\} \setminus \varphi$ . Using this description we define

$$M(\Delta) = \{(\sigma_1, \dots, \sigma_d) \in \{0, \pm 1\}^d : (|\sigma_1|, \dots, |\sigma_d|) \in \Delta\},$$

the **mirror complex** of  $\Delta$ , which is a subcomplex of the  $d$ -cube.

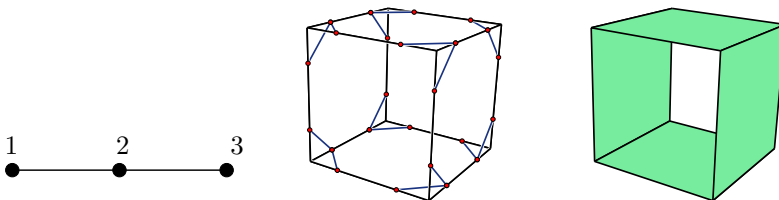


Figure 2. Simplicial complex  $\Delta$  on three vertices, its symmetric embedding into  $C_d$ , and the mirror complex  $M(\Delta)$ .

By construction the vertex figure of the mirror complex of a simplicial complex is isomorphic to the simplicial complex itself. Thus if the simplicial complex is a  $d$ -sphere, then its mirror complex is a  $(d + 1)$ -manifold. The  $f$ -vector of the mirror complex of a simplicial complex  $\Delta$  on  $n$  vertices is  $f_i(M(\Delta)) = 2^{n-i} f_{i-1}(\Delta)$ . Further the mirror complex of the boundary of a simplicial complex is the boundary of the mirror complex:  $M(\partial\Delta) = \partial M(\Delta)$ .

**PROPOSITION 2.1:** *Let  $\Delta$  be a simplicial complex with automorphism group  $\Gamma$ . Then the automorphism group of  $M(\Delta)$  is isomorphic to the semi-direct product  $\Sigma_d \rtimes \Gamma$ .*

**2.4. FISSURING.** The cubical fissure or fissuring is an operation that produces a new cubical complex from a given one. Let  $C$  be a pure cubical  $d$ -complex,  $C_1$  and  $C_2$  facet-disjoint  $d$ -dimensional subcomplexes of  $C$  such that  $C_1 \cup C_2 = C$ . The **cubical fissure**  $\text{fis}_C(C_1, C_2)$  of  $C$  between  $C_1$  and  $C_2$  is defined by lifting  $C_1$  to height one, dropping  $C_2$  to height minus one and filling in the fissure with  $(C_1 \cap C_2) \times [-1, 1]$ . The corresponding poset is

$$\text{fis}_C(C_1, C_2) = (C_1 \times \{+1\}) \cup ((C_1 \cap C_2) \times \{0\}) \cup (C_2 \times \{-1\}),$$

with  $+1 < 0$  and  $-1 < 0$  in the last component. If  $C$  is a subcomplex of the  $n$ -cube given as sign vectors of length  $n$ , then the cubical fissure canonically yields a subcomplex of the  $(n + 1)$ -cube. The cubical fissure between a subcomplex  $C_1$  and its complement  $C_2$  yields a complex consisting of the subcomplex  $C_1$  and its complement connected via a prism over the boundary of  $C_1$ .

*Example 2.2:* Consider the simplicial complex  $\Delta$  of Figure 2 on three vertices. The mirror complex  $C_1 = M(\Delta)$  is a subcomplex of the boundary complex of the 3-cube  $C = \partial C_3$ . So the cubical fissure between  $C_1$  and its complement in  $C$  is a subcomplex of the boundary of the 4-cube, see Figure 3.

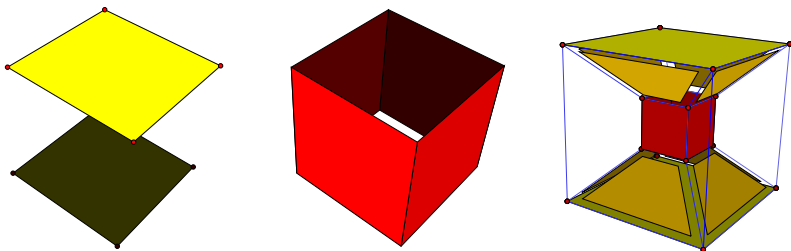


Figure 3. Two subcomplexes of the 3-cube and their cubical fissure in the Schlegel diagram of the 4-cube.

2.5. BBC SEQUENCES OF SIMPLICIAL BALLS. Babson, Billera, and Chan [4] proved the existence of neighborly cubical spheres. Their approach is based on an inductive construction using triangulations of cyclic polytopes, mirror complexes and fissuring. A close inspection of their proof motivates the following definition. While this does look a bit technical it is one of the keys to our main results.

*Definition 2.3:* Let  $V = \{v_1, \dots, v_n\}$  be a set of vertices and let  $\mathcal{T} = (T_i)_{i=d+1}^n$  be a sequence of simplicial  $d$ -balls. We say that  $\mathcal{T}$  is a **BBC sequence** if

- (i) each  $T_i$  is of the form  $T_i = B_{i-1} * v_i$ , where  $B_{i-1}$  is a simplicial  $(d-1)$ -ball on the vertex set  $\{v_1, \dots, v_{i-1}\}$ ; and
- (ii)  $B_{i-1} \subseteq \partial T_{i-1}$  for  $i = d+2, \dots, n$ .

*Remark 2.4:* Let  $P$  be a simplicial  $(d+1)$ -polytope with vertices  $v_1, \dots, v_n$ . We may assume that the vertices of  $P$  lie in general position since  $P$  is simplicial. So every subset of at least  $d+2$  of its vertices is again the vertex set of a simplicial  $(d+1)$ -polytope. Thus by taking an arbitrary ordering  $(v_{l_1}, \dots, v_{l_n})$  of the vertices of  $P$  we obtain a sequence of  $(d+1)$ -balls  $P_i = \text{conv}(v_{l_1}, \dots, v_{l_i})$  for  $i = d+2, \dots, n$ . The corresponding BBC sequence  $(T_i)_{i=d+2}^n$  is the sequence of pulling triangulations of  $P_i$  with respect to  $v_{l_i}$ .

*Example 2.5:* Consider the pentagon with its vertices labeled in cyclic order and construct a BBC sequence as in Remark 2.4. The BBC sequence consists of three elements: the triangle  $T_3$ , the pulling triangulations of the 4-gon  $T_4$ , and the pulling triangulation of the 5-gon  $T_5$ . The vector representation of  $T_i$  and  $\partial T_i$  for  $i = 3, 4, 5$  are given in Table 1. The table also shows the subcomplexes  $B_3 \subset \partial T_3$  and  $B_4 \subset \partial T_4$  such that  $T_4 = B_3 * v_4$  and  $T_5 = B_4 * v_5$ .

Table 1. BBC sequence of a pentagon as described in Example 2.5. The vectors in  $\{0, 1\}^i$  are the characteristic functions of the complement of the vertex sets in  $[i]$ , for example 11001 represents  $\{3, 4\}$ . The subcomplex  $B_{i-1}$  of  $\partial T_{i-1}$  such that  $T_i$  is the join of  $B_{i-1}$  with the vertex  $v_i$  are printed in bold.

	Facets of $T_i$			Facets of $\partial T_i$				
$T_3$	000			<b>001</b>	<b>100</b>	010		
$T_4$	0010	1000		<b>0011</b>	<b>1001</b>	<b>1100</b>	0110	
$T_5$	00110	10010	11000	00111	10011	11001	11001	01110

Note that  $(T_i)_{i=d+1}^n$  is a BBC sequence if and only if each boundary sphere  $\partial T_i$  is **directly obtainable** from its predecessor  $\partial T_{i-1}$  in the sense of Altshuler [1]. In loc. cit. he proved that there is a 3-sphere on ten vertices which is *not* directly obtainable. Remark 2.4 implies that such a sphere cannot be polytopal.

*Remark 2.6:* Since each sphere of dimension at most 2 is polytopal it follows that every BBC sequence of 3-balls is a sequence of pulling triangulations of polytopes. But there exist BBC sequences of simplicial 4-balls such that the final boundary sphere is not polytopal. One such sequence is described in Section 3.5, Table 4.

**2.6. NEIGHBORLY BBC SEQUENCES.** A simplicial complex is **(simplicially)  $k$ -neighborly** if every  $k$ -subset of its vertices is a face. In other words, its  $(k - 1)$ -skeleton is isomorphic to the  $(k - 1)$ -skeleton of a simplex. A simplicial  $d$ -sphere is **(simplicially) neighborly** if it is simplicially  $\lfloor (d+1)/2 \rfloor$ -neighborly. In the following  $[k]$  denotes the set of positive integers  $\{1, \dots, k\}$ .

The next two observations are also implicit in the work of Shemer [23]. We start off with a lemma connecting the neighborliness of the cone over a simplicial ball with the neighborliness of the simplicial ball.

**LEMMA 2.7:** *Let  $B$  be a simplicial  $(d - 1)$ -ball on the vertex set  $[i - 1]$ , and let  $T = B * i$  its  $d$ -dimensional cone for some  $d \geq 2$ . Then  $\partial T$  is  $k$ -neighborly if and only if  $\partial B$  is  $(k - 1)$ -neighborly and  $B$  is  $k$ -neighborly.*

*Proof:* The boundary  $\partial T$  is equal to  $(\partial B * i) \cup B$  since  $T = B * i$ . If  $\partial T$  is  $k$ -neighborly all  $k$ -subsets of  $[i]$  are faces of  $\partial T$ . Because  $\partial T$  is the union of  $\partial B * i$  and  $B$ , all  $(k - 1)$ -subsets of  $[i - 1]$  must be contained in  $\partial B$  and  $B$  must contain all  $k$ -subsets of  $[i - 1]$ . This means that  $\partial B$  is  $(k - 1)$ -neighborly and  $B$  is  $k$ -neighborly.

Conversely, if  $\partial B$  is  $(k - 1)$ -neighborly and  $B$  is  $k$ -neighborly then  $\partial T = (\partial B * i) \cup B$  obviously contains all the  $k$ -subsets of  $[i]$ . ■

We call a BBC sequence  $(T_i)_{i=d+1}^n$  **neighborly** if the final boundary sphere  $\partial T_n$  is neighborly. With Lemma 2.7 and an induction we obtain a characterization of neighborly BBC sequences.

PROPOSITION 2.8: *Let  $(T_i)_{i=d+1}^n$  be a BBC sequence of  $d$ -balls for  $d \geq 2$  with  $B_i$  defined as in Definition 2.3. Then the following are equivalent:*

- (1)  $(T_i)_{i=d+1}^n$  is a neighborly BBC sequence.
- (2)  $\partial T_i$  is a neighborly  $(d - 1)$ -sphere for all  $i = d + 1, \dots, n$ .
- (3) For all  $i = d + 1, \dots, n$  the ball  $B_{i-1}$  is  $\lfloor d/2 \rfloor$ -neighborly, and the sphere  $\partial B_{i-1}$  is  $(\lfloor d/2 \rfloor - 1)$ -neighborly.

Neighborly BBC sequences arise naturally from neighborly simplicial polytopes as in Remark 2.4. So the above definition is a generalization of the sequences of pulling triangulations of cyclic polytopes originally used by Babson, Billera, and Chan.

COROLLARY 2.9: *Take an arbitrary ordering of the vertices of a neighborly simplicial polytope. Then there exists a realization such that the induced pulling triangulations form a neighborly BBC sequence.*

### 3. Cubical Spheres from BBC sequences

In Section 3.1 we generalize the construction of Babson, Billera, and Chan using BBC sequences. Then, in Section 3.2, we derive a non-recursive sign vector representation for the cubical sphere constructed from a BBC sequence. Then we show that the neighborly cubical spheres built from special vertex orderings of cyclic polytopes are indeed isomorphic to the neighborly cubical polytopes studied in [14]. Further we construct a non-polytopal neighborly cubical sphere based on a BBC sequence obtained from the non-polytopal Altshuler 3-sphere on 10 vertices (cf. Altshuler [2] and Bokowski and Garms [7]).

3.1. CUBICAL SPHERES. Let  $\mathcal{T} = (T_i)_{i=d}^{n-1}$  be a BBC sequence of simplicial  $(d - 1)$ -balls. We inductively define cubical complexes  $S_{k+1}$  for  $k = d, \dots, n - 1$  with  $2^{k+1}$  vertices. Babson, Billera, and Chan begin their inductive definition with  $S_d$ , which is two  $d$ -cubes identified at their complete boundary. But since this does not yield a regular cell complex, we start with  $S_{d+1} = \partial C_{d+1}$  the boundary of the  $(d + 1)$ -cube (that is, the mirror complex of the boundary of the  $d$ -simplex). Then for  $k = d + 1, \dots, n - 1$  we recursively define

$$\begin{aligned} S_{k+1} &= \text{fis}_{S_k}(\text{M}(T_k), S_k \setminus \text{M}(T_k)) \\ &= (\text{M}(T_k) \times \{+1\}) \cup (\partial \text{M}(T_k) \times \{0\}) \cup ((S_k \setminus \text{M}(T_k)) \times \{-1\}). \end{aligned}$$

We denote the final cubical complex  $S_n$  by  $\text{bbc}(\mathcal{T})$ .



**THEOREM 3.1:** *Let  $\mathcal{T} = (T_i)_{i=d}^{n-1}$  be a BBC sequence of simplicial  $(d - 1)$ -balls. Then the cubical complexes  $S_k$  and, in particular,  $\text{bbc}(\mathcal{T})$  are cubical  $d$ -spheres.*

*Proof:* This is part of what is proved by Babson, Billera and Chan [4, Theorem 3.1]. The essential step in the inductive argument is to see that the mirror complex of  $T_k$  is actually a subcomplex of  $S_k$ , such that fissuring is possible.

If the simplicial balls  $T_i$  are equipped with a piecewise-linear (PL) structure, then the resulting cubical spheres are also PL [4].

**3.2. A COMBINATORIAL DESCRIPTION.** The following theorem states a purely combinatorial description of the cubical spheres constructed from BBC sequences. Its characterization as a subcomplex of a high dimensional cube is close to the Cubical Gale Evenness Condition [14, Theorem 18].

**THEOREM 3.2:** *Let  $\mathcal{T} = (T_i)_{i=d}^{n-1}$  be a BBC sequence of simplicial  $(d - 1)$ -balls with  $n > d > 2$ . Then the facets of the cubical  $d$ -sphere  $\text{bbc}(\mathcal{T})$  correspond to the following list of  $d$ -faces of the  $n$ -dimensional cube  $C_n$ ; they are represented as sign vectors  $\alpha \in \{0, \pm 1\}^n$  with exactly  $d$  zeros. The type  $t$  of a facet corresponds to the number of trailing non-zero entries in  $\alpha$ :*

- ▷ **type  $t = 0$ :**  $\alpha_n = 0$  and  $|\alpha^{(n-1)}| := (|\alpha_1|, \dots, |\alpha_{n-1}|) \in \partial T_{n-1}$ ,
- ▷ **type  $0 < t < n - d$ :**  $\alpha = (\alpha^{(n-t-1)}, 0, \alpha_{n-t+1} = \sigma, -1, \dots, -1)$ , where  $\sigma \in \{\pm 1\}$ ,  $\alpha^{(n-t-1)} \in \{0, \pm 1\}^{n-t-1}$  with
  - (1)  $|\alpha^{(n-t-1)}| \in \partial T_{n-t-1}$ , and
  - (2) if  $\sigma = +1$ , then  $|\alpha^{(n-t-1)}, 0| \in T_{n-t}$ ;  
 if  $\sigma = -1$ , then  $|\alpha^{(n-t-1)}, 0| \notin T_{n-t}$ ,
- ▷ **type  $t = n - d$ :**  $\alpha = (0, \dots, 0, \sigma, -1, \dots, -1)$  with  $\sigma \in \{-1, +1\}$ .

*Proof:* We prove that the vector representation of the facets of the cubical sphere given by the theorem corresponds to the facets of the inductive definition of  $S_k$  used in Theorem 3.1:

$$S_{k+1} = (M(T_k) \times \{+1\}) \cup (\partial M(T_k) \times \{0\}) \cup ((S_k \setminus M(T_k)) \times \{-1\})$$

It was already shown in Theorem 3.1 that all  $S_k$  are cubical spheres. We proceed by induction on  $k$ . The sphere  $S_{d+1}$  is the boundary of the  $(d + 1)$ -cube for  $k = d + 1$ . The facets of  $S_{d+1}$  are the vectors  $\alpha \in \{0, \pm 1\}^{d+1}$  with exactly one non-zero entry. The facets of  $T_d = \Delta_{d-1}$  are the vectors in  $\{0, 1\}^d$  with

one 1. Thus all the facets of  $S_{d+1}$  are either of type 0 or type 1 shown in Table 2.

Table 2. The facets of  $S_{d+1}$  are the facets of the  $(d + 1)$ -cube.

Type		Facets		
0	$\pm$	0	$\dots$	0
0	0	$\pm$	0	$\vdots$
0	$\vdots$	$\ddots$	$\ddots$	0
1	0	$\dots$	0	$\pm$

The inductive step is split into two claims showing that the vector description and the inductive definition of  $S_k$  yield the same combinatorics.

CLAIM: *Each facet in the vector representation given by the theorem is also a facet in the inductive definition via fissuring.*

We analyze all types of facets of the vector representation of  $S_{k+1}$ .

▷ **type 0:** The facets of type 0 are the vectors  $\alpha = (\alpha^{(k)}, 0) \in \{0, \pm 1\}^{k+1}$  with  $|\alpha^{(k)}| \in \partial T_k$ . This is equivalent to  $\alpha^{(k)} \in M(\partial T_k)$  and thus  $(\alpha^{(k)}, 0) \in M(\partial T_k) \times \{0\} = \partial M(T_k) \times \{0\}$ .

▷ **type 1:** The facets of type 1 are the vectors  $\alpha = (\alpha^{(k-1)}, 0, \sigma)$  with  $|\alpha^{(k-1)}| \in \partial T_{k-1}$  and

- (1)  $\sigma = +1$  if  $(|\alpha^{(k-1)}|, 0) \in T_k$  or
- (2)  $\sigma = -1$  if  $(|\alpha^{(k-1)}|, 0) \notin T_k$ .

By induction  $(\alpha^{(k-1)}, 0)$  is a type 0 facet of  $S_k$ , and

- (1) if  $(|\alpha^{(k-1)}|, 0) \in T_k$  then  $(\alpha^{(k-1)}, 0) \in M(T_k)$  and thus  $(\alpha^{(k-1)}, 0, +1)$  is contained in  $M(T_k) \times \{+1\} \subseteq S_{k+1}$ .
- (2) Otherwise, if  $(|\alpha^{(k-1)}|, 0) \notin T_k$  then  $(\alpha^{(k-1)}, 0) \in S_k \setminus M(T_k)$  and thus  $(\alpha^{(k-1)}, 0, -1) \in (S_k \setminus M(T_k)) \times \{-1\} \subseteq S_{k+1}$ .

▷ **type 1**  $1 < t_0 \leq k + 1 - d$ : The facets of type  $t_0$  are the vectors

$$(\alpha^{(k-t_0)}, 0, \sigma, -1, \dots, -1) \in \{0, \pm 1\}^{k+1}$$

with  $\sigma \in \{\pm 1\}$  and  $\alpha^{(k-t_0)} \in \{0, \pm 1\}^{k-t_0}$ . Taking only the first  $k$  entries of  $\alpha = (\alpha^{(k)}, -1)$ , it follows by induction that  $\alpha^{(k)}$  is a type  $t_0 - 1$  facet of  $S_k$ . By the definition  $T_k$  is a cone with apex  $k$  and thus all its facets contain the vertex  $k$ . Since the last entry of  $\alpha^{(k)}$  is not zero,  $\alpha^{(k)}$  cannot be a facet of  $M(T_k)$ . Further it is not contained in  $M(\partial T_k)$  because it has  $d$  zero entries. Thus  $\alpha^{(k)}$  is a facet of  $S_k \setminus M(T_k)$  and  $(\alpha^{(k)}, -1) \in S_{k+1}$ .

CLAIM: *Each facet in the inductive definition via fissuring is also a facet in the vector representation given by the theorem.*

There are three different kinds of facets of  $S_{k+1}$  according to the inductive definition. We will determine the type of each kind of facet.

- ▷ The facets of  $M(T_k) \times \{+1\}$  are the vectors  $\alpha = (\alpha^{(k)}, +1) \in \{0, \pm 1\}^{k+1}$  with  $|\alpha^{(k)}| \in T_k$ . Since by definition of the BBC sequence  $T_k = B_{k-1} * v_k$  with  $B_{k-1} \subseteq \partial T_{k-1}$ , we obtain  $\alpha_k^{(k)} = 0$  and  $|\alpha^{(k-1)}| \in \partial T_{k-1}$ . Hence  $\alpha$  is a facet of type 1 of  $S_{k+1}$ .
- ▷ The facets of  $M(\partial T_k) \times \{0\}$  are the vectors  $(\alpha^{(k)}, 0) \in \{0, \pm 1\}^{k+1}$  with  $|\alpha^{(k)}| \in \partial T_k$ . These are the facets of type 0 of  $S_{k+1}$ .
- ▷ The facets of  $(S_k \setminus M(T_k)) \times \{-1\}$  correspond to vectors  $\alpha = (\alpha^{(k)}, -1)$  with  $\alpha^{(k)} \in S_k \setminus M(T_k)$ . Using the inductive assumptions we distinguish two kinds of facets of  $S_k$ :

- (1) Either  $\alpha^{(k)}$  is a facet of type 0 of  $S_k$  not contained in  $M(T_k)$ , that is,  $\alpha^{(k)} = (\alpha^{(k-1)}, 0)$  with  $|\alpha^{(k-1)}| \in \partial T_{k-1}$  and  $\alpha^{(k)} \notin M(T_k)$ ,
- (2) or  $\alpha^{(k)}$  is a facet of type  $t_0 \in \{1, \dots, k-d\}$  of  $S_k$  not contained in  $M(T_k)$ , that is,  $\alpha^{(k)} = (\alpha^{(k-1)}, \pm 1)$ .

In the first case,  $\alpha$  is a facet of type 1 of  $S_{k+1}$  since  $(|\alpha^{(k-1)}|, 0) \notin T_k$ . In the second case,  $\alpha$  is a facet of type  $t_0 + 1$  of  $S_{k+1}$ .

Hence the cubical sphere  $\text{bbc}((T_i)_{i=d}^{n-1}) = S_n$  has the facets given by the theorem. ■

If  $\mathcal{T} = (T_i)_{i=d}^{n-1}$  is a neighborly BBC sequence,  $n > d > 2$ , then each simplicial  $(d-2)$ -sphere  $\partial T_i$  is neighborly. The number of facets  $s(i, d-2)$  of  $\partial T_i$  is determined by the Dehn–Sommerville equations, see Grünbaum [12, §9.2]. For odd dimension  $d$  we have

$$s(i, d-2) = \binom{i - \frac{d-1}{2}}{i+1-d} + \binom{i - \frac{d+1}{2}}{i+1-d} = \frac{2i}{2i-d+1} \binom{i - \frac{d-1}{2}}{i+1-d}.$$

Adding up these equations (according to the types of facets described in Theorem 3.2) results in a formula for  $f(n, d)$ , the number of facets of the (neighborly) cubical sphere  $\text{bbc}(\mathcal{T})$ . The even dimensional case may be treated similarly and is left to the reader. The same formula occurs in [14, Corollary 19] where, however, the even and the odd-dimensional case are erroneously exchanged:

COROLLARY 3.3: *Let  $\mathcal{T} = (T_i)_{i=d}^{n-1}$  be a neighborly BBC sequence of simplicial  $(d-1)$ -balls with  $n > d > 2$  and  $d$  odd. Then the number of facets of  $\text{bbc}(\mathcal{T})$*

is given by

$$\begin{aligned}
 f(n, d) &= 2(d + 1) + \sum_{t=0}^{n-d-2} s(n - t - 1, d - 2)2^{n-t-d} \\
 &= 2(d + 1) + \sum_{k=d+2}^n \frac{2k - 2}{2k - 1 - d} \binom{k - \frac{d+1}{2}}{k - d} 2^{k-d}.
 \end{aligned}$$

A notion of a neighborly cubical complex is defined in Section 3.3.

*Example 3.4:* A (neighborly) cubical 3-sphere with the graph of the 6-cube may be constructed from a BBC sequence of the pentagon described in Example 2.5. The  $f$ -vector of the sphere is  $(64, 192, 192, 64)$ . The vector representations of  $T_i$  and  $\partial T_i$  for  $i = 3, 4, 5$  are given in Table 1. The mirror complexes of  $T_i$  and  $\partial T_i$  are obtained by simply replacing the 1's by  $\pm 1$ 's. So according to the inductive definition we start with  $S_4$ , the boundary of the 4-cube and fissure first along  $M(\partial T_4)$  and then along  $M(\partial T_5)$ . This yields the facet description of  $S_6$  listed in Table 3.

Table 3. Facets of the neighborly cubical 3-sphere  $S_6$  sorted by type; see Example 3.4. The facets occur in different multiplicities depending on the number of  $\pm$  entries in the corresponding row: For example, every row of type 0 represents eight facets. The rows of the  $4 \times 4$  minor printed in bold are the facets of  $S_4$ , that is, the boundary of the 4-cube. The facets of  $S_5$  are the  $8 \times 5$  minor printed in bold and plain. The horizontal lines underline the inductive structure.

Type	Facets					
3	<b>0</b>	<b>0</b>	<b>0</b>	$\pm$	$-$	$-$
2	<b>0</b>	<b>0</b>	$\pm$	<b>0</b>	$+$	$-$
2	<b>0</b>	$\pm$	<b>0</b>	<b>0</b>	$-$	$-$
2	$\pm$	<b>0</b>	<b>0</b>	<b>0</b>	$+$	$-$
1	0	0	$\pm$	$\pm$	0	$+$
1	$\pm$	0	0	$\pm$	0	$+$
1	$\pm$	$\pm$	0	0	0	$+$
1	0	$\pm$	$\pm$	0	0	$-$
0	<b>0</b>	<b>0</b>	$\pm$	$\pm$	$\pm$	<b>0</b>
0	$\pm$	<b>0</b>	<b>0</b>	$\pm$	$\pm$	<b>0</b>
0	$\pm$	$\pm$	<b>0</b>	<b>0</b>	$\pm$	<b>0</b>
0	$\pm$	$\pm$	$\pm$	<b>0</b>	<b>0</b>	<b>0</b>
0	<b>0</b>	$\pm$	$\pm$	$\pm$	<b>0</b>	<b>0</b>

3.3. NEIGHBORLY CUBICAL SPHERES. A cubical complex is **(cubically)  $k$ -neighborly** if it has the  $(k - 1)$ -skeleton of a cube. A cubical  $d$ -sphere is **(cubically) neighborly** if it is cubically  $\lfloor (d+1)/2 \rfloor$ -neighborly. This is similar to the simplicial case and not as in Babson, Billera and Chan, where a  $k$ -neighborly cubical sphere has the  $k$ -skeleton of a cube.

The neighborliness of a simplicial complex is preserved by mirroring, that is, the mirror complex of a  $k$ -neighborly simplicial complex is a  $(k + 1)$ -neighborly cubical complex: In its sign vector representation a  $k$ -neighborly simplicial complex  $\Delta$  on  $n$  vertices contains all vectors in  $\{0, 1\}^n$  with  $k$  zeros, i.e. all  $k$ -subsets of  $[n]$ . Thus its mirror complex  $M(\Delta)$  contains all vectors in  $\{0, \pm 1\}^n$  with  $k$  zeros. These vectors represent the  $k$ -skeleton of the  $n$ -cube, which means  $M(\Delta)$  is  $(k + 1)$ -neighborly.

Cubical neighborliness is also preserved by certain fissuring operations. Given a  $k$ -neighborly pure cubical complex  $C$  and a subcomplex  $S$  containing all vertices. If the boundary of  $S$  is  $k$ -neighborly, then the cubical fissure between  $S$  and its complement in  $C$  is again a  $k$ -neighborly cubical complex.

This yields neighborly cubical spheres from any neighborly BBC sequence with the same construction as in Theorem 3.1.

**COROLLARY 3.5:** *Let  $\mathcal{T}$  be a neighborly BBC sequence of simplicial  $(d - 1)$ -balls. Then the cubical complex  $\text{bbc}(\mathcal{T})$  is a neighborly cubical  $d$ -sphere.*

3.4. NEIGHBORLY CUBICAL POLYTOPES. In this section we show that for very particular neighborly BBC sequences, the neighborly cubical spheres constructed therefrom are isomorphic to the boundaries of the neighborly cubical polytopes described in [14, Theorem 18] and thus polytopal. The neighborly cubical polytopes have the following sign vector representation.

**THEOREM 3.6 (Cubical Gale Evenness Condition):** *The facets of the neighborly cubical polytope  $\text{ncp}_{d+1}(n)$  are given by vectors  $\alpha \in \{0, \pm 1\}^n$  with  $d$  zeros. They are classified by the number  $t$  of leading  $\pm 1$ 's:*

- ▷ **type  $t = 0$ :**  $\alpha_1 = 0$ , and  $|\alpha|$  satisfies the simplicial Gale Evenness Condition: between any two values  $\alpha_i, \alpha_j \in \{\pm 1\}$  there is an even number of zeros.

- ▷ **type**  $0 < t < n - d$ :  $\alpha = (-1, +1, \dots, (-1)^{t-1}, \sigma, 0, \alpha^{(n-t-1)})$ , with  $\sigma \in \{\pm 1\}$ ,  $\alpha^{(n-t-1)} \in \{0, \pm 1\}^{n-t-1}$  and:
  - (1)  $|\alpha^{(n-t-1)}|$  satisfies the simplicial Gale Evenness Condition, and
  - (2) if  $\sigma = (-1)^{t+1}$ , then  $\alpha^{(n-t-1)}$  starts with an even number of zeros; if  $\sigma = (-1)^t$ , then  $\alpha^{(n-t-1)}$  starts with an odd number of zeros.
- ▷ **type**  $t = n - d$ :  $\alpha = (-1, +1, \dots, (-1)^{t-1}, \sigma, 0, \dots, 0)$  with  $\sigma \in \{-1, +1\}$ .

Consider the cyclic  $(d - 1)$ -polytope on  $n$  vertices  $\text{cyc}_{d-1}(n)$  given as the convex hull of the vertices  $v_j = (j, j^2, \dots, j^{d-1})$  for  $j = 1, \dots, n$ . The facets of the cyclic polytope are given by Gale’s Evenness Condition, cf. [10] and [25, Theorem 0.7]. Let  $T_i$  denote the pulling triangulation of  $\text{cyc}_{d-1}(i)$  with respect to the vertex  $v_i$ . The pulling triangulation is the cone with apex  $v_i$  over the facets of  $\text{cyc}_{d-1}(i)$  not containing  $v_i$ . To satisfy Gale’s Evenness Condition the facets not containing  $v_i$  have to end with an even number of zeros. Thus the facets of  $T_i$  according to our vector notation for simplicial complexes are the vectors  $\varphi \in \{0, 1\}^i$  with  $d$  zeros, such that:

- (1)  $\varphi_i = 0$ , i.e. all facets contain  $v_i$ ,
- (2)  $(\varphi_1, \dots, \varphi_{i-1})$  satisfies Gale’s Evenness Condition, and
- (3)  $\varphi$  ends with an odd number of zeros.

The facets of the neighborly cubical sphere constructed from the corresponding neighborly BBC sequence  $(T_i)_{i=d}^{n-1}$  are readily derived from Theorem 3.2. These specific spheres are polytopal, since they are isomorphic to the neighborly cubical polytopes.

**COROLLARY 3.7:** *Let  $d \geq 3$  and  $T_i$  be the pulling triangulation of the cyclic polytope  $\text{cyc}_{d-1}(i)$  with respect to the last vertex as above. Then the neighborly cubical sphere  $\text{bbc}((T_i)_{i=d}^{n-1})$  and the boundary of the neighborly cubical polytope  $\text{ncp}_{d+1}(n)$  are combinatorially isomorphic. The isomorphism is given by inverting the order and then flipping the even bits:*

$$\Phi: \{0, \pm 1\}^n \rightarrow \{0, \pm 1\}^n,$$

$$(\alpha_1, \alpha_2, \dots, \alpha_n) \mapsto (\alpha_n, -\alpha_{n-1}, \dots, (-1)^n \alpha_2, (-1)^{n+1} \alpha_1).$$

*Remark 3.8:* We briefly summarize some results studied in [22, Section 3.6.3]. The neighborly cubical  $d$ -spheres constructed from BBC sequences of cyclic  $(d - 1)$ -polytopes on  $d + 1$  vertices are combinatorially isomorphic.

Since in **even** dimension the automorphism group of every cyclic polytope acts transitively on its vertices, its pulling triangulations are combinatorially isomorphic. Thus the neighborly cubical spheres constructed from arbitrary vertex orderings of even dimensional cyclic polytopes are combinatorially unique and thus all polytopal. This includes Example 3.4.

In **odd** dimension  $d - 1$ , the automorphism group of a cyclic polytope with more than  $d + 1$  vertices does not act transitively on the vertices. Thus we were able to construct non-isomorphic neighborly cubical 4-spheres from different triangulations of the cyclic 3-polytope on six vertices.

**3.5. A NON-POLYTOPAL NEIGHBORLY CUBICAL SPHERE.** To algorithmically decide the polytopality of spheres is generally possible but of considerable complexity, see Bokowski and Sturmfels [8], Richter-Gebert [19], and Basu et al. [5]. If standard heuristic methods fail the problem can often be approached by applying suitable ad-hoc techniques only. A good example is the 3-sphere  $\mathcal{M}_{425}^{10}$  found by Altshuler in an exhaustive enumeration of all neighborly simplicial 3-manifolds on ten vertices [2]. Its polytopality could not be decided at first, and so it was up to Bokowski and Garms [7] to prove that  $\mathcal{M}_{425}^{10}$  does not admit any convex realization. Below we take this very example as the starting point for a construction of a neighborly cubical sphere which cannot be polytopal in view of the following result.

**THEOREM 3.9:** *Let  $\mathcal{T} = (T_i)_{i=d}^{n-1}$  be a BBC sequence of simplicial  $(d - 1)$ -balls, neighborly or not, such that the cubical  $d$ -sphere  $\text{bbc}(\mathcal{T})$  is polytopal. Then the simplicial  $(d - 2)$ -sphere  $\partial T_{n-1}$  is necessarily polytopal.*

*Proof:* Suppose that  $\text{bbc}(\mathcal{T})$  is isomorphic to the boundary complex of a convex cubical  $(d + 1)$ -polytope  $P \subset \mathbb{R}^{d+1}$ . Consider the edge  $e$  of  $P$  corresponding to the sign vector  $(+1, \dots, +1, 0)$  of length  $n$ . Choose an affine hyperplane  $H$ , parallel to  $e$ , which separates the edge  $e$  from the  $2^n - 2$  vertices of  $P$  not contained in  $e$ . Next we choose a second affine hyperplane  $H'$ , orthogonal to  $e$ , which separates the two vertices of  $e$ . Then, since  $H$  and  $H'$  are not parallel to each other,  $P/e = P \cap H \cap H'$  is a  $(d - 1)$ -dimensional convex polytope, the **edge figure** of  $e$  with respect to  $P$ . Its face lattice is isomorphic to the filter of  $e$ , that is, the faces of  $P$  containing  $e$ , in the face lattice of  $P$ .

Now the facets of  $P$  which contain  $e$  are exactly the facets of type 0 without negative signs in their sign vector representation. From Theorem 3.2 we conclude that  $\partial(P/e)$  is isomorphic to  $\partial T_{n-1}$ , and hence the claim. ■

As a consequence, each BBC sequence of simplicial  $(d - 1)$ -balls with the property that its final boundary sphere is non-polytopal yields a non-polytopal cubical sphere. Note, however, that in contrast to polytopal spheres, see Corollary 2.9, for non-polytopal spheres there is no standard procedure to obtain a corresponding BBC sequence. Moreover, it follows from work of Altshuler that there is a simplicial 3-sphere on ten vertices which does not admit any BBC sequence [1].

**THEOREM 3.10:** *There is a non-polytopal neighborly cubical 5-sphere with  $2^{11} = 2048$  vertices and  $f(11, 5) = 3584$  facets. Its complete  $f$ -vector is*

$$f = (2048, 11264, 28160, 33280, 17920, 3584).$$

*Proof:* Table 4 lists a neighborly BBC sequence  $\mathcal{A} = (A_i)_{i=5}^{10}$  of simplicial 4-balls with the property that the boundary of the final ball  $A_{10}$  is isomorphic to the Altshuler 3-sphere  $\mathcal{M}_{425}^{10}$ .

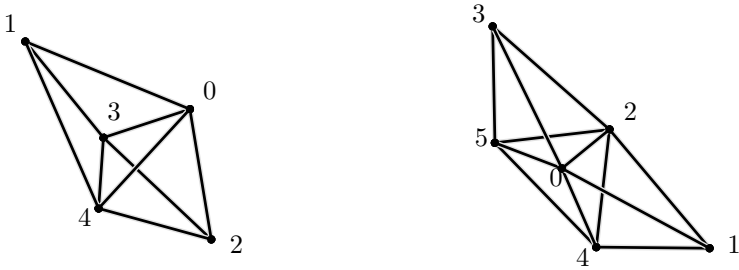
The simplicial 4-ball  $A_8$  is a pulling triangulation of the neighborly simplicial 4-polytope on eight vertices which occurs as  $P_{36}^8$  in the list of Grünbaum and Sreedharan [13]; this is one of the two non-cyclic neighborly simplicial polytopes with these parameters.

The simplicial 4-ball  $A_9$  is a pulling triangulation of a neighborly simplicial 4-polytope on nine vertices: The previous simplicial 3-sphere  $\partial A_8$  is separated by the 2-sphere  $\partial B_8$  into the 3-balls  $B_8$  and its complement  $B'_8$ . The boundary  $\partial A_9$  now is directly obtained from  $\partial A_8$  by first removing  $B'_8$  and then inserting the cone over  $\partial B_8$  with the new vertex 9 as the apex. Since  $\partial A_8$  is polytopal and  $B'_8$  is a 3-simplex which is stacked once over each of its four facets it follows that  $A_9$  is again polytopal: The vertex 9 can be chosen in a way such that it is **exactly beyond** all the facets of  $B'_8$  in the sense of Shemer [23, page 301].

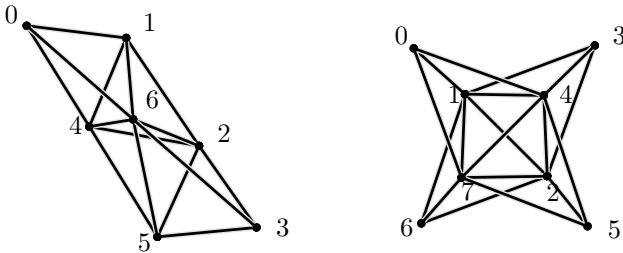
The sphere  $\partial A_{10}$  is directly obtained from  $\partial A_9$  and isomorphic to  $\mathcal{M}_{425}^{10}$  and thus not polytopal due to Bokowski and Garms [7]. From Theorem 3.9 it follows that the neighborly cubical sphere  $\text{bbc}(T)$  is not polytopal. ■



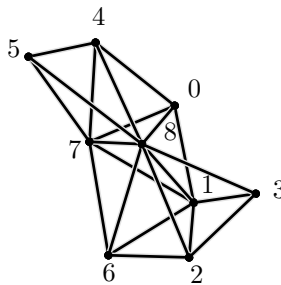
Figure 4 visualizes the complements of the simplicial 3-balls  $B_5, B_6, B_7, B_8,$  and  $B_9$  and their boundaries.



(a) Bipyramid over triangle:  $\partial B_5$ . (b) 3-simplex stacked on two facets:  $\partial B_6$ .



(c) Cone over a strip of four triangles:  $\partial B_7$ . (d) 3-simplex stacked on all its facets:  $\partial B_8$ .



Cone over a strip of six triangles:  $\partial B_9$ .

Figure 4. The subcomplexes  $B'_i = \partial A_i \setminus B_i$  of the BBC sequence of the Altshuler sphere; see Theorem 3.10. The sphere  $\partial B_i = \partial B'_i$  is the link of the new vertex  $i$  in the sphere  $\partial A_{i+1}$ .

*Remark 3.11:* The polytopal neighborly simplicial 3-sphere  $\partial A_9$  is the unique 3-sphere on nine vertices with Altshuler determinant equal to 103361328. It occurs as  $N_{19}^9$  in the classification of Altshuler and Steinberg [3] and as `manifold_3_9_523` in Lutz [15].

Table 4. Facets of the neighborly BBC sequence  $\mathcal{A}$  of the non-polytopal Altshuler sphere  $\mathcal{M}_{425}^{10}$ . The simplicial ball  $A_i$  is the join of the vertex  $i - 1$  and the subcomplex  $B_{i-1}$  of the facets of  $\partial A_{i-1}$  printed in bold.

	Facets of $A_i$				Facets of $\partial A_i$				
$A_5$	01234				<b>0123</b>	<b>0124</b>	0134	0234	<b>1234</b>
$A_6$	01235	01245	12345		<b>0123</b>	0124	<b>0135</b>	<b>0145</b>	0235
					0245	<b>1234</b>	<b>1345</b>	<b>2345</b>	
$A_7$	01236	01356	01456	12346	<b>0123</b>	<b>0126</b>	<b>0135</b>	<b>0145</b>	0146
	13456	23456			<b>0236</b>	<b>0356</b>	<b>0456</b>	<b>1234</b>	1246
					<b>1345</b>	<b>2345</b>	2356	2456	
$A_8$	01237	01267	01357	01457	<b>0123</b>	<b>0126</b>	<b>0135</b>	<b>0145</b>	0147
	02367	03567	04567	12347	<b>0167</b>	<b>0236</b>	<b>0356</b>	<b>0456</b>	<b>0467</b>
	13457	23457			1234	1247	1267	<b>1345</b>	<b>2345</b>
					<b>2357</b>	<b>2367</b>	2457	<b>3567</b>	<b>4567</b>
$A_9$	01238	01268	01358	01458	<b>0123</b>	<b>0126</b>	<b>0135</b>	<b>0145</b>	<b>0148</b>
	01678	02368	03568	04568	<b>0167</b>	0178	<b>0236</b>	<b>0356</b>	<b>0456</b>
	04678	13458	23458	23578	<b>0467</b>	0478	1238	1268	<b>1345</b>
	23678	35678	45678		<b>1348</b>	1678	<b>2345</b>	<b>2348</b>	<b>2357</b>
					<b>2367</b>	<b>2458</b>	<b>2578</b>	<b>2678</b>	<b>3567</b>
					<b>4567</b>	4578			
$A_{10}$	01239	01269	01359	01459	0123	0126	0135	0145	0148
	01489	01679	02369	03569	0167	0179	0189	0236	0356
	04569	04679	13459	13489	0456	0467	0479	0489	1239
	23459	23489	23579	23679	1269	1345	1348	1389	1679
	24589	25789	26789	35679	2345	2348	2357	2367	2389
	45679				2458	2578	2678	2689	3567
					4567	4579	4589	5789	6789

### 4. Polyhedral Surfaces

In this section we describe a nice way to realize polyhedral surfaces of ‘unusually large genus’ [17] in the Schlegel diagram of neighborly cubical polytopes. The surfaces considered are cubical polyhedral surfaces where each vertex has degree  $q$ . They were first described by Coxeter [9] in 1937 in terms of reflection groups. Ringel [20] explicitly described these surfaces whilst analyzing problems concerning the graph of the  $n$ -dimensional cube. He pointed out that the surfaces are of lowest genus among all surfaces on which the graph of the  $n$ -cube may be drawn without self intersection. Further he gave an explicit combinato-

rial description of these surfaces as a 2-dimensional subcomplexes of the  $n$ -cube. McMullen, Schulz and Wills [16, 17] analyze equivelar surfaces, a much more general class of polyhedral surfaces which include the cubical surfaces of Coxeter and Ringel.

*Definition 4.1:* An **equivelar surface**  $\mathcal{M}_{p,q}$  is a polyhedral surface such that all 2-faces are  $p$ -gons and all vertices have degree  $q$ .

McMullen, Schulz and Wills were the first to point at the ‘unusually large genus’ of these surfaces. In particular, they inductively constructed embeddings for the cubical equivelar surfaces of type  $\mathcal{M}_{4,q}$  in  $\mathbb{R}^3$ .

4.1. EQUIVELAR SURFACES AND MIRROR COMPLEXES. Let  $Q$  be the boundary of a  $q$ -gon for  $q > 2$  and  $M(Q)$  its mirror complex. With the vertices of  $Q$  labelled in cyclic order we obtain the same vector representation as Ringel [20, page 17] of  $M(Q)$ :

$$\begin{array}{cccccccc}
 0 & 0 & \pm & \pm & \cdots & \pm & \pm & \pm \\
 \pm & 0 & 0 & \pm & \cdots & \pm & \pm & \pm \\
 \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
 \pm & \pm & \pm & \pm & \cdots & \pm & 0 & 0 \\
 0 & \pm & \pm & \pm & \cdots & \pm & \pm & 0
 \end{array}$$

The mirror complex of  $Q$  is an equivelar surface  $\mathcal{M}_{4,q}$  since the vertex figure of every vertex of  $M(Q)$  is isomorphic to  $Q$ . It is embedded in the 2-skeleton of the  $q$ -cube.

The obvious way to realize a 2-dimensional subcomplex of the  $q$ -cube in  $\mathbb{R}^5$  is in the Schlegel diagram of the 6-dimensional neighborly cubical polytope  $n\text{cp}_6(q)$  of [14]: Since the neighborly cubical polytope  $n\text{cp}_6(q)$  has the 2-skeleton of the  $q$ -cube,  $M(Q)$  is contained in the boundary of  $n\text{cp}_6(q)$ . The Schlegel diagram of  $n\text{cp}_6(q)$  is embedded in  $\mathbb{R}^5$ , thus  $M(Q)$  may be realized in  $\mathbb{R}^5$ .

Let  $T_i$  be the pulling triangulation of the  $i$ -gon for  $i = 3, \dots, q - 1$ . The facets of type 0 of  $\mathcal{S}_3(q) = \text{bbc}((T_i)_{i=3}^{q-1})$  are the mirror complex of a cone over the boundary of the  $(q - 1)$ -gon. Since the  $q$ -gon and its pulling triangulation are subcomplexes of this cone, the mirror complex of the  $q$ -gon and the mirror complex of its pulling triangulation are both subcomplexes of  $\mathcal{S}_3(q)$ . This yields a cubificated embedding of the surface  $M(Q)$  into the neighborly cubical 3-sphere  $\mathcal{S}_3(q)$ . By Corollary 3.7 this sphere is isomorphic to the boundary of the neighborly cubical polytope  $n\text{cp}_4(q)$ . Hence the mirror complex of the  $q$ -gon can be realized as a subcomplex of the Schlegel diagram of  $n\text{cp}_4(q)$  in  $\mathbb{R}^3$ .

This answers a question of Günter M. Ziegler [24] whether some of the surfaces from [17] can be found as subcomplexes of the neighborly cubical polytopes.

The genus of this surface may easily be calculated from its  $f$ -vector  $f(M(Q)) = (2^q, 2^{q-1}q, 2^{q-2}q)$ :

$$g(q) = 1 + 2^{q-3}(q - 4) = \mathcal{O}(f_0 \cdot \log f_0).$$

Thus  $q = 12$  is the first parameter for which the genus  $g(12) = 4097$  exceeds the number of vertices, which equals  $2^{12} = 4096$ .

Since the surface arising from the 12-gon is too hard to visualize we display the mirror complex of the pentagon in the Schlegel diagram of  $\text{ncp}_4(5)$  in Figure 5.

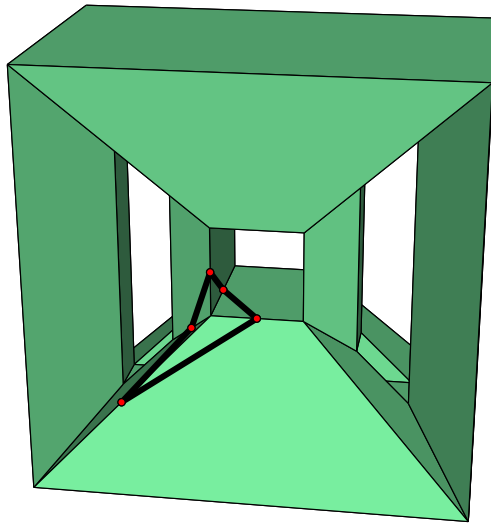


Figure 5. The mirror complex of the 5-gon is an equivelar surface of type  $\mathcal{M}_{4,5}$  of genus 5. It is realized in the Schlegel diagram of the neighborly cubical polytope  $\text{ncp}_4(5)$ , which is embedded in  $\mathbb{R}^3$ .

### 5. Concluding Remarks and Acknowledgments

It can be shown that if the BBC sequence  $\mathcal{T}$  is polytopal then the cubical sphere  $\text{bbc}(\mathcal{T})$  is also polytopal [21]. In view of Theorem 3.9 this gives the complete picture of the construction as far as questions of polytopality are concerned.

For the visualization of the simplicial balls and the polyhedral surface  $\mathcal{M}_{4,5}$  we used the software packages `polymake` [11] and `JavaView` [18].

We are indebted to Frank H. Lutz for his help in recognizing the sphere  $\partial A_9$  in Table 4. Moreover, we are grateful to an anonymous referee who scrutinized the submitted version of this paper and who pointed out several minor errors.

### References

- [1] A. Altshuler, *A peculiar triangulation of the 3-sphere*, Proceedings of the American Mathematical Society **54** (1976), 449–452.
- [2] A. Altshuler, *Neighborly 4-polytopes and neighborly combinatorial 3-manifolds with ten vertices*, Canadian Journal of Mathematics **29** (1977), 400–420.
- [3] A. Altshuler and L. Steinberg, *Neighborly 4-polytopes with 9 vertices*, Journal of Combinatorial Theory, Series A **15** (1973), 270–287.
- [4] E. K. Babson, L. J. Billera and C. S. Chan, *Neighborly cubical spheres and a cubical lower bound conjecture*, Israel Journal of Mathematics **102** (1997), 297–315.
- [5] S. Basu, R. Pollack and M.-F. Roy, *Algorithms in Real Algebraic Geometry*, Algorithms and Computation in Mathematics, Vol. 10, Springer-Verlag, Berlin, 2003.
- [6] A. Björner, *Posets, regular CW complexes and Bruhat order*, European Journal of Combinatorics **5** (1984), 7–16.
- [7] J. Bokowski and K. Garms, *Altshuler’s sphere  $M_{425}^{10}$  is not polytopal*, European Journal of Combinatorics **8** (1987), 227–229.
- [8] J. Bokowski and B. Sturmfels, *Computational Synthetic Geometry*, Lecture Notes in Mathematics, vol. 1355, Springer-Verlag, Berlin, 1989.
- [9] H. S. M. Coxeter, *Regular skew polyhedra in three and four dimensions, and their topological analogues*, Proceedings of the London Mathematical Society, II. Ser. **43** (1937), 33–62.
- [10] D. Gale, *Neighborly and cyclic polytopes*, in *Proceedings of Symposia in Pure Mathematics, Vol. VII*, American Mathematical Society, Providence, R.I., 1963, pp. 225–232.
- [11] E. Gawrilow and M. Joswig, *polymake: a framework for analyzing convex polytopes*, in *Polytopes—combinatorics and computation* (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73.  
<http://www.math.tu-berlin.de/polymake>.
- [12] B. Grünbaum, *Convex Polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003, Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.
- [13] B. Grünbaum and V. P. Sreedharan, *An enumeration of simplicial 4-polytopes with 8 vertices*, Journal of Combinatorial Theory **2** (1967), 437–465.

- [14] M. Joswig and G. M. Ziegler, *Neighborly cubical polytopes*, Discrete and Computational Geometry **24** (2000), 325–344, The Branko Grünbaum birthday issue.
- [15] F. H. Lutz, *The Manifold Page, 1999–2005*,  
<http://www.math.tu-berlin.de/diskregeom/stellar/>.
- [16] P. McMullen, C. Schulz and J. M. Wills, *Equivelar polyhedral manifolds in  $E^3$* , Israel Journal of Mathematics **41** (1982), 331–346.
- [17] P. McMullen, C. Schulz and J. M. Wills, *Polyhedral 2-manifolds in  $E^3$  with unusually large genus*, Israel Journal of Mathematics **46** (1983), 127–144.
- [18] K. Polthier, K. Hildebrandt, E. Preuss and U. Reitebuch, *JavaView v3.03*, [www.javaview.de](http://www.javaview.de), 2004.
- [19] J. Richter-Gebert, *Realization Spaces of Polytopes*, Lecture Notes in Mathematics, vol. 1643, Springer-Verlag, Berlin, 1996.
- [20] G. Ringel, *Über drei kombinatorische Probleme am  $n$ -dimensionalen Würfel und Würfelgitter*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **20** (1955), 10–19.
- [21] R. Sanyal, T. Rörig and G. M. Ziegler, *Polytopes and Polyhedral Surfaces via Projection*, in preparation.
- [22] T. Rörig, *Neighborly cubical spheres and polytopes*, Master's thesis, Technische Universität Berlin, 2004, <http://www.math.tu-berlin.de/tilosch/>.
- [23] I. Shemer, *Neighborly polytopes*, Israel Journal of Mathematics **43** (1982), 291–314.
- [24] G. M. Ziegler, *personal communication*, March 2004.
- [25] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.